New Parallel Algorithms for Direct Solution of Sparse Linear Systems : Part I - Symmetric Coefficient Matrix

Kartik Gopalan C. Siva Ram Murthy *

Department of Computer Science and Engineering
Indian Institute of Technology
Madras 600 036, India
e-mail: murthy@iitm.ernet.in

Abstract

In this paper, we propose a new parallel *bidirectional algorithm*, based on *Cholesky factorization*, for the solution of sparse *symmetric* system of linear equations. Unlike the existing algorithms, the numerical factorization phase of our algorithm is carried out in such a manner that the entire back substitution component of the substitution phase is replaced by a single step division. Since there is a substantial reduction in the time taken by the repeated execution of the substitution phase, our algorithm is particularly suited for the solution of systems with multiple b-vectors. The effectiveness of our algorithm is demonstrated by comparing it with the existing parallel algorithm, based on Cholesky factorization, using extensive simulation studies.

In Part II of this paper, we propose a new parallel *bidirectional algorithm*, based on *LU factorization*, for the solution of general sparse system of linear equations, having *non-symmetric* coefficient matrix.

Key Words: Linear Equation, Sparse Symmetric System, Cholesky Factorization, Parallel Algorithm, Bidirectional Scheme, Multiprocessor.

1 Introduction

The problem of solving large *sparse systems of linear equations* arises in various applications such as finite element analysis, power system analysis, and circuit simulation for VLSI CAD.

*Author for correspondence.

In this paper, we consider the problem of solving sparse symmetric system of linear equations of the form $Ax = b$, where A is a sparse symmetric coefficient matrix of dimension $N \times N$, and x and b are N-vectors. Traditionally, the process for obtaining the direct solution for a sparse symmetric system of linear equations, $Ax = b$, involves the following four distinct phases.

- *Ordering* : Apply an appropriate symmetric permutation matrix P such that the new system is of the form $(PAP^T)(Px) = (Pb)$.
- *Symbolic factorization* : Set up the appropriate data structures for the numerical factorization phase.
- *Numerical factorization* : Determine the Cholesky factor L such that $A = LL^T$.
- *Substitution* : Determine the solution vector x by first solving the forward triangular system $Ly = b$ and then solving the backward triangular system $L^T x = y$.

For solution of multiple b-vectors, the first three phases are carried out only once to obtain the Cholesky factor L. The substitution phase is then repeated for each b-vector in order to obtain a different solution vector x in each case. Thus, in problems which involve solution of multiple b-vectors, the time taken by repeated execution of substitution phase dominates the overall solution time. Any parallel formulation, which can reduce the time taken by the substitution phase, will contribute significantly to enhanced performance of the entire process.

Although traditional approaches to parallel solution of sparse symmetric system of linear equations have yielded efficient parallel algorithms for the numerical factorization phase [1, 2, 7, 11, 15, 21], not much progress has been made in the case of substitution phase due to the limited amount of parallelism inherent in this phase. Moreover, the forward and backward substitution components of the substitution phase require different parallel algorithms due to the manner in which data is distributed over various processors. Existing work on parallel formulations for this phase can be found in [6, 12, 14].

In Part I of this paper, we present a new *bidirectional algorithm*, based on Cholesky factorization, for the solution of sparse symmetric system of linear equations. In our algorithm, the numerical factorization phase is carried out in such a manner that the entire back substitution component of the substitution phase is replaced by a single step division. The bidirectional algorithm, based on *LU factorization*, for the solution of general sparse system of linear equations, is presented in Part II of this paper. The application of the novel concept of bidirectional elimination to dense linear systems can be found in [19, 20].

The rest of the paper is organized as follows. In section 2, we present the bidirectional sparse Cholesky factorization algorithm for sparse symmetric matrices. In section 3, we present the bidirectional algorithm for the substitution phase which does not have a back substitution component. In section 4, we develop a bidirectional heuristic algorithm for ordering on the lines of the popular *nested dissection* ordering algorithm [4, 5] for sparse symmetric matrices. In section 5, we describe a symbolic factorization algorithm which sets up data structures required by the bidirectional Cholesky factorization phase. In section 6, we evaluate the performance of the bidirectional algorithm on hypercube multiprocessors and present comparison of our algorithm with the existing scheme based on sparse Cholesky factorization. In section 7, we conclude this paper with some observations about possible future improvements to the bidirectional scheme.

2 The Bidirectional Sparse Cholesky Factorization (B-SCF) Algorithm

Unlike the regular Cholesky factorization algorithm which factorizes A to obtain the lower triangular matrix L, such that $A = LL^T$, the BSCF algorithm factorizes A into a series of *trapezoidal* matrices of multipliers. This series of trapezoidal matrices remove the need for the back substitution component in the substitution phase.

In this section, we first present an overall view of the concept of bidirectional Cholesky factorization. We then proceed to describe the manner in which the sparsity of the coefficient matrix can be exploited to obtain higher degree of parallelism. Following this we present the details of implementing BSCF algorithm on multiprocessor systems.

2.1 Bidirectional Cholesky Factorization - The Concept

In regular Cholesky algorithm, the lower triangular matrix L is obtained by choosing columns 1 through N of matrix A as pivots so that $A = LL^T$. We name this process as *factorization in forward direction.* On the other hand, we can also choose columns N through 1 of matrix A as pivots and factorize A in a reverse fashion to obtain an upper triangular matrix U such that $A = U^TU$. We name this process as *factorization in backward direction.* The bidirectional Cholesky factorization of the coefficient matrix A proceeds as follows.

- *Step 1*: We form two matrices, namely A_0 and A_1, identical to the coefficient matrix A. We factorize A_0 in the forward direction, but only through the first $\lceil \frac{N}{2} \rceil$ pivot columns, to obtain a lower trapezoidal matrix L_0, as shown in figure 1, in which only

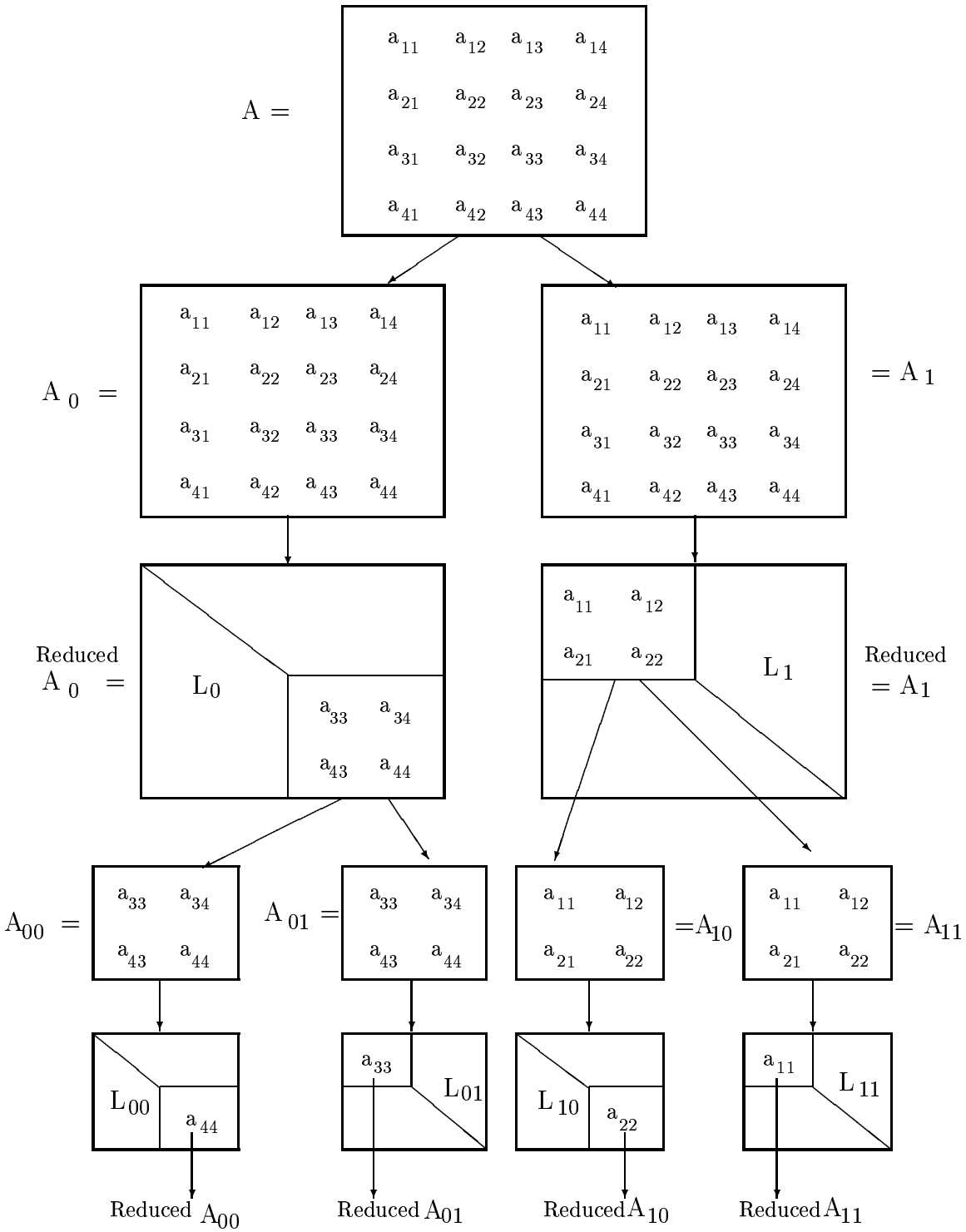


Figure 1: The progression of BSCF algorithm for $N = 4$

the sub-diagonal entries in columns 1 to $\lceil \frac{N}{2} \rceil$ are present. Concurrently, we factorize A_1 in backward direction, through pivot columns N to $\lceil \frac{N}{2} + 1 \rceil$, to obtain an upper trapezoidal matrix L_1, as shown in figure 1, in which only the super-diagonal elements in columns N to $\lceil \frac{N}{2} + 1 \rceil$ are present.

- *Step 2*: We duplicate the reduced matrix A_0 to form A_{00} and A_{01}, and also duplicate the reduced matrix A_1 to form A_{10} and A_{11}. The matrices A_{00} and A_{10} are factorized halfway through in the forward direction to produce lower trapezoidal matrices L_{00} and L_{10} respectively. Similarly, the matrices A_{01} and A_{11} are factorized halfway through in the backward direction to produce upper trapezoidal matrices L_{01} and L_{11} respectively. Note that here we factorize the four matrices A_{00}, A_{01}, A_{10}, and A_{11} in parallel.

- *Step 3*: We continue this process of halving the size of the sub-matrices through simultaneous Cholesky factorization in both forward and backward directions and thus doubling the number of sub-matrices for $\log N$ times. Finally we end up with N sub-matrices of order 1×1.

The bidirectional Cholesky factorization algorithm described above produces a tree of trapezoids of multipliers (i.e., L matrices). In the substitution phase, which is described in section 3, the b-vector, corresponding to which a solution vector x has to be found, is moved down this tree of trapezoids. At the end of this process each leaf produces an equation with just one variable which is then solved by a single step division to produce the solution vector x.

2.2 Exploiting the Sparsity of the Coefficient Matrix A

In regular sparse Cholesky factorization of a coefficient matrix A, column i directly modifies column j if $j > i$ and $A[i, j] \neq 0$. Column i indirectly modifies column j if column i directly modifies another column k which in turn modifies column j directly or indirectly. Columns i and j are *mutually independent* if column i does not modify column j directly or indirectly. The mutually independent columns of the sparse matrix can be used as pivots in parallel.

This concept of mutually independent columns can be easily extended to the BSCF algorithm. At any stage $s \in \{1 \cdots \log N\}$, columns i and j ($j > i$) are *forward independent* if pivot column i does not modify column j directly or indirectly during factorization in forward direction. The forward independent columns, i and j, can be simultaneously used as pivots in forward direction. The columns i and j are *backward independent* if pivot column j does not modify column i directly or indirectly during factorization in backward

direction. The backward independent columns, i and j, can be simultaneously used as pivots in backward direction.

In regular sparse Cholesky factorization, the concept of mutually independent columns can be abstracted with the help of *elimination trees*. An elimination tree contains a node corresponding to each column of the coefficient matrix. The parent of a node i is defined as

$$parent(i) = min\{j \mid j > i \text{ and } L[j,i] \neq 0\}.$$

The elimination tree defines a partially ordered precedence relation which determines when a certain column can be used as pivot.

Similarly, in BSCF algorithm, we can abstract the concepts of forward independence and backward independence by means of *forward elimination tree* and *backward elimination tree* respectively. At some stage $s \in \{1 \cdots \log N\}$, let A_{x0} be a sub-matrix being factorized in the forward direction and A_{x1} be a sub-matrix being factorized in the backward direction (x being a possibly empty string of 0's and 1's). The *forward parent* of node i, is defined as

$$fparent(i, A_{x0}) = min\{j \mid j > i \text{ and } L_{x0}[j,i] \neq 0\}.$$

Similarly, the *backward parent* of node i, is defined as

$$bparent(i, A_{x1}) = max\{j \mid j < i \text{ and } L_{x1}[j,i] \neq 0\}.$$

For achieving high degree of parallelism during factorization phase, both the forward and the backward elimination trees should be as short and wide as possible. This is the function of the ordering phase (described in section 4).

In the next subsection, we examine the parallel implementation of BSCF algorithm on multiprocessors.

2.3 Implementing the BSCF Algorithm on Multiprocessors

For our present study, we consider the *medium grain model* of parallelism in which tasks perform floating point operations over nonzero elements of entire columns of coefficient matrix [18]. The following elementary tasks are considered for the BSCF algorithm.

- *fdivide(i,s)* divides by $\sqrt{A_{x0}[i,i]}$ every nonzero element of the sub-diagonal part of the ith column of sub-matrix A_{x0}.
- *bdivide(i,s)* divides by $\sqrt{A_{x1}[i,i]}$ every nonzero element of the super-diagonal part of the ith column of sub-matrix A_{x1}.

- *fmodify(i,vector,s)* subtracts the contents of *vector* from the ith column of a sub-matrix A_{x0}, at stage $s \in \{1 \cdots \log N\}$. *vector* is an appropriate multiple of some column j of A_{x0}, which modifies column i directly in forward direction at stage s.

- *bmodify(i,vector,s)* subtracts the contents of *vector* from the ith column of a sub-matrix A_{x1}, at stage $s \in \{1 \cdots \log N\}$. *vector* is an appropriate multiple of some column j of A_{x1}, which modifies column i directly in backward direction at stage s.

To keep track of the columns that each pivot should modify at each of the $\log N$ stages, we maintain the following data structures.

- $F_i^{(s)}$ denotes the set of all columns with indices smaller than i that modify the ith column in the forward direction at stage s.

- $B_i^{(s)}$ denotes the set of all columns with indices greater than i that modify the ith column in the backward direction at stage s.

These data structures are generated during the symbolic factorization phase. This phase is described in section 5. In the remaining part of this section, we describe the implementation of BSCF algorithm on a message passing multiprocessor

In Cholesky factorization, if column i modifies column j, then the factor, by which the modifying column i is multiplied, is an element $A[j,i]$ of the modifying column i itself. This happens due to the symmetric nature of the coefficient matrix being operated upon. Thus, the multiple of the modifying column can be calculated at the processor storing column i itself and the resulting vector can be sent over to the processor storing column j which needs to be modified.

When $p < N$, there might be more than one column at a processor P_k, which modifies column j (i.e., more than one column stored at processor P_k might belong to the sets $F_j^{(s)}$ or $B_j^{(s)}$). In place of sending a separate vector as message corresponding to every column at P_k that modifies column j, we can add all these outgoing vectors together and send them as one vector to the processor storing column j. In this manner, the number of outgoing messages can be significantly reduced. Note that the above observation applies for modifications in both forward and backward factorizations.

In algorithm 1 below, we incorporate the above idea in the BSCF algorithm and present the *fan-in* BSCF algorithm. The set $List_{myid}$ is the set of columns stored in processor P_{myid}. Each processor maintains the sparse vectors $fUpdate_j$ and $bUpdate_j$ for $1 \leq j \leq N$. If column i is to modify column j in forward direction at stage s then, after performing $fdivide(i,s)$ operation, the processor P_{myid}, which stores the column i, adds an appropriate

multiple of column i to the vector $fUpdate_j$. When such an addition has been performed for all the columns in processor P_{myid} that modify column j in forward direction at stage s, a message containing the $fUpdate$ vector is sent to the processor storing the column j. Similar mechanism operates for factorization in backward direction.

Algorithm 1 (*The parallel fan-in BSCF algorithm)

begin

for $s := 1$ **to** $\log N$ **do**

parbegin

Forward_factorize($List_{myid}$,s);

Backward_factorize($List_{myid}$,s);

parend

end

procedure Forward_factorize($List$,s)

begin

for $i := 0$ **to** $N-1$ **do** $fUpdate_i$:=**0**;

while $List \neq \phi$ **do**

if $\exists i \in List$ such that $fdivide(j,s)$ has been performed for all $j \in F_i^{(s)}$ **then**

Let column i belong to the forward sub-matrix A_{x0} at stage s;

while messages of the form (i,$fvector$,s) have not been received from

all processors that store columns belonging to $F_i^{(s)}$ **do**

receive messages of the form (i,$fvector$,s);

$fmodify(i, fvector, s)$;

if column i belongs to the first half of sub-matrix A_{x0} **then**

$fdivide(i,s)$;

for all j such that $i \in F_j^{(s)}$ **do**

$fUpdate_j := fUpdate_j + A_{x0}[j,i] \times A_{x0}[*,i]$;

if $fdivide(k,s)$ has been done for all $k \in F_k^{(s)} \cap List$ **then**

send a message of the form (j,$fUpdate_j$,s)

to processor storing column j;

else if $s < \log N$ **then**

(*copy column i of A_{x0} to column i of A_{x00}*)

$A_{x00}[*,i] := A_{x0}[*,i]$;

(*copy column i of A_{x0} to row i of A_{x01} since only sub-diagonal

parts of the columns of the symmetric matrix A_{x0} are stored*)
for all j such that $A_{x0}[j,i] \neq 0$ **do**
$A_{x01}[i,j] := A_{x0}[j,i]$;
$List := List - i$;
end

procedure Backward_factorize($List$,s)
begin
for $i := 0$ **to** $N-1$ **do** $bUpdate_i$:=**0**;
while $List \neq \phi$ **do**
if $\exists i \in List$ such that $bdivide(j,s)$ has been performed for all $j \in B_i^{(s)}$ **then**
Let column i belong to the backward sub-matrix A_{x1} at stage s;
while messages of the form (i,$bvector$,s) have not been received from
all processors that store columns belonging to $B_i^{(s)}$ **do**
receive messages of the form (i,$bvector$,s);
$bmodify(i, bvector, s)$;

if column i belongs to the second half of sub-matrix A_{x1} **then**
$bdivide(i,s)$;
for all j such that $i \in B_j^{(s)}$ **do**
$bUpdate_j := bUpdate_j + A_{x1}[j,i] \times A_{x1}[*,i]$;
if $bdivide(k,s)$ has been done for all $k \in B_k^{(s)} \cap List$ **then**
send a message of the form (j,$bUpdate_j$,s)
to processor storing column j;

else if $s < \log N$ **then**
(*copy column i of A_{x1} to row i of A_{x10} since only super-diagonal
part of the columns of the symmetric matrix A_{x1} are stored*)
for all j such that $A_{x1}[j,i] \neq 0$ **do**
$A_{x10}[i,j] := A_{x1}[j,i]$;
(*copy column i of A_{x1} to column i of A_{x11}*)
$A_{x11}[*,i] := A_{x1}[*,i]$;
$List := List - i$;
end

An important observation is in order in algorithm 1. Let the number of processors $p = 2^d$ (as in hypercube multiprocessors) and $N = 2^n$ ($n, d \in \mathcal{N}$, the set of natural numbers).

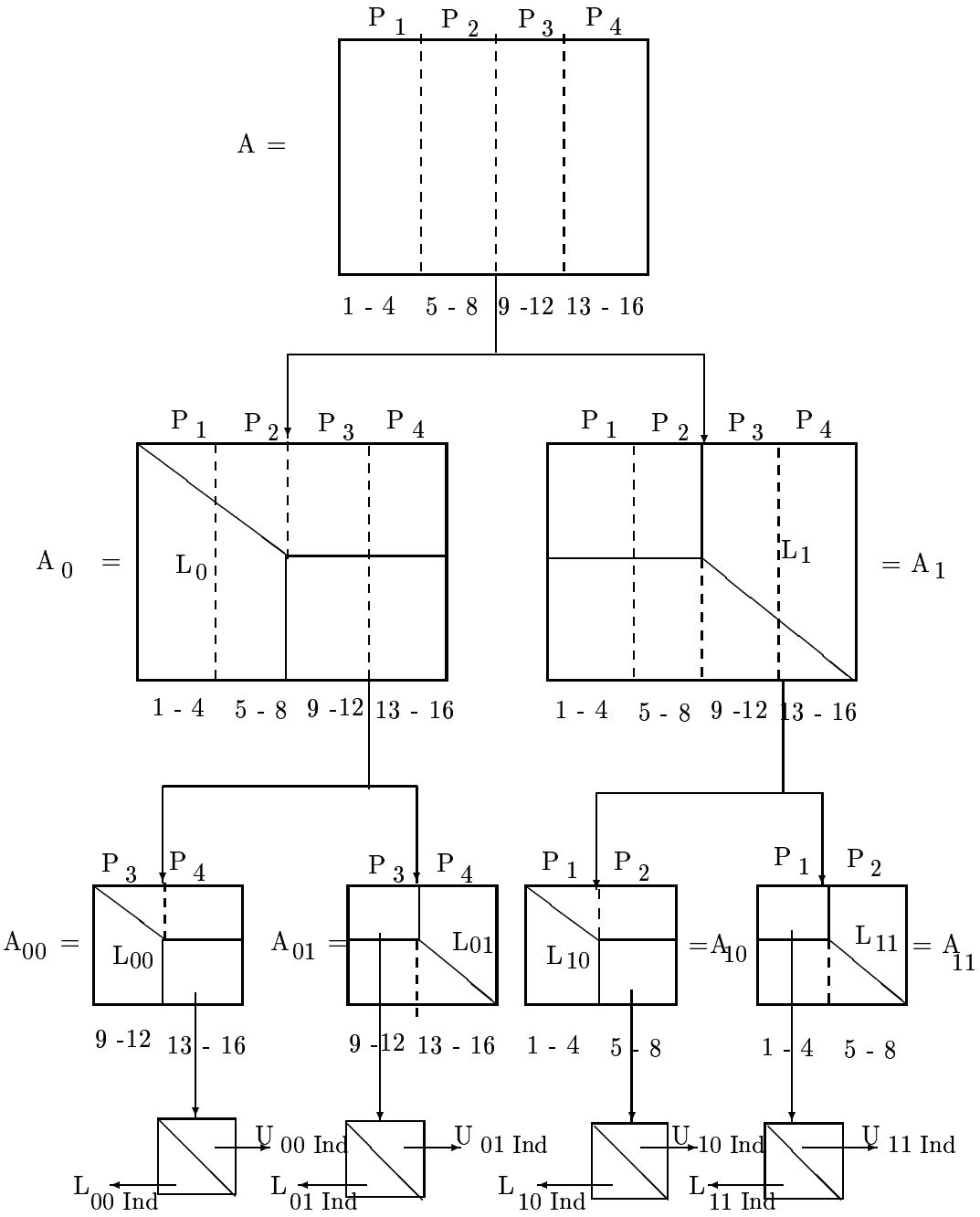


Figure 2: Progression of the BSCF algorithm for $p = 4$ and $N = 16$ (four columns are stored in each processor).

Assume that we map the equations on the processors in a block wrap manner (as shown in figure 2). Thus each processor holds $\frac{N}{p} = 2^{n-d}$ consecutive equations. At the end of $d = \log p$ stages of the fan-in BSCF algorithm, each processor contains an independent system of $\frac{N}{p}$ equations. This independent system can be factorized within a single processor without any communication with any other processor. Since, on a single processor, regular sequential sparse Cholesky factorization performs more efficiently than the fan-in BSCF algorithm, we can switch over to this regular sequential version after $\log p$ stages and factorize the coefficient matrix (say A_{ind}) of this independent system into the form $A_{ind} = L_{ind}L_{ind}^T$. This results in enhancing the performance of the fan-in BSCF algorithm. The manner in which this factorization proceeds is shown in figure 2.

In this figure, we also note that the size of the subset of processors with which any processor P_i communicates, reduces by half with every stage. In stage $s = 1$, all processors P_1 through P_4 communicate with each other. In stage $s = 2$, P_1 and P_2 communicate only with each other, and P_3 and P_4 communicate only with each other. Thus communication gets localized with every stage.

3 The Substitution Phase

In this section we present the *bidirectional substitution* (BS) algorithm. Unlike the regular algorithm, which consists of two triangular solution components (i.e., the forward substitution followed by the backward substiution), the BS algorithm consists of only one forward solution component, which is followed by a single step division to yield the solution vector x. Following the pattern of the previous section, we first present an overall view of the concepts behind the BS algorithm. We then proceed to describe the manner in which the sparsity of the series of trapezoidal factor matrices can be exploited to obtain a higher degree of parallelism.

3.1 Bidirectional Substitution Algorithm - The Concept

The scheme we propose below is somewhat on similar lines to the parallel column triangular solver (PCTS) proposed by Li and Coleman in [17]. To find the solution vector x, for a given b-vector, we begin with two copies of b-vectors b_0 and b_1.

- *Step 1* : The vector b_0 is modified by successive columns of trapezoids of multipliers L_0 (i.e., from column 1 to column $\lceil \frac{N}{2} \rceil$). In other words, after modification by column $i-1$, the processor containing column i computes x_i as $x_i = b_0[i]/L_0[i,i]$ and modifies

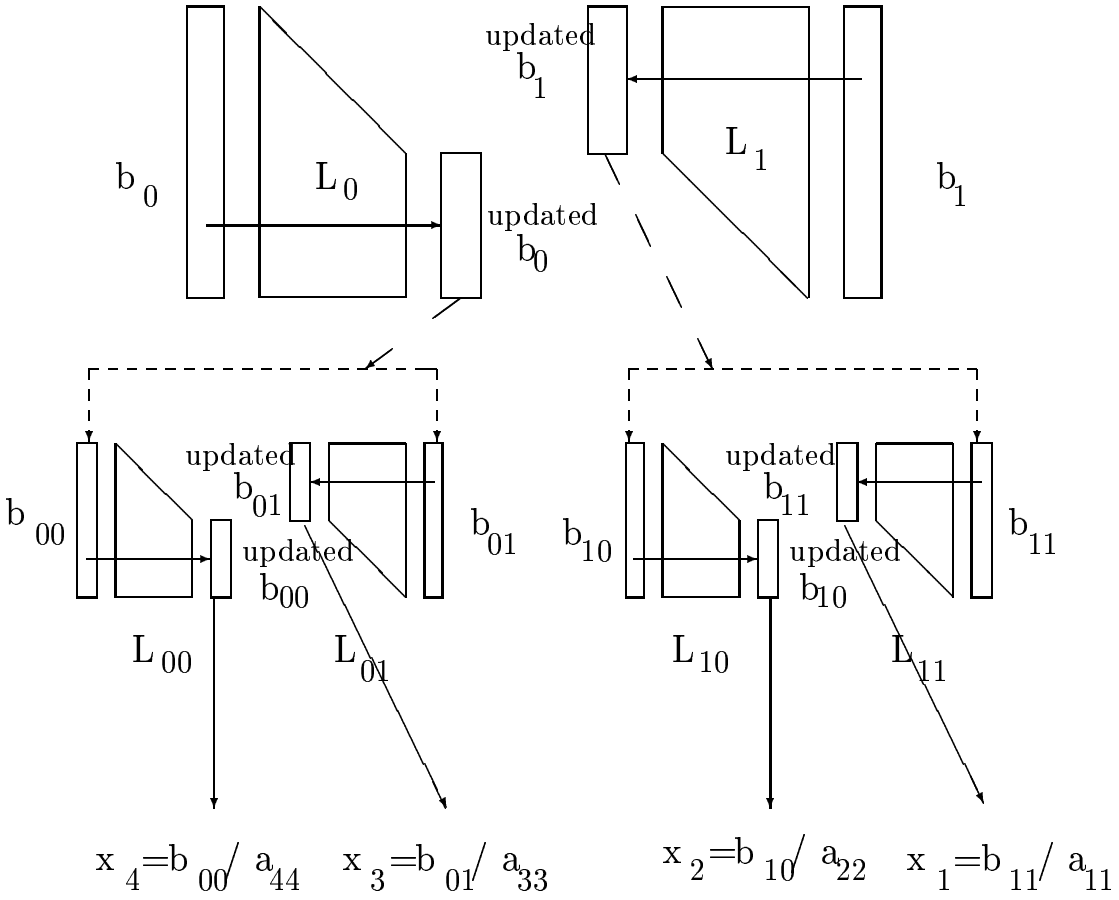


Figure 3: The progression of substitution phase for $N = 4$

the remaining elements of b_0-vector as $b_0[j] = b_0[j] - L_0[j,i] * x_i$ for all j such that $L_0[j,i] \neq 0$. At the end of updation by L_0, the size of vector b_0 is reduced to half its original size (see figure 3). Simultaneously, the vector b_1 is updated by successive columns of the trapezoidal matrix of multipliers L_1 in backward direction (i.e., from column N to column $\lceil \frac{N}{2} \rceil + 1$). In other words, after modification by column $i+1$, the processor containing column i computes x_i as $x_i = b_1[i]/L_1[i,i]$ and modifies the remaining b_1-vector as $b_1[j] = b_1[j] - L_1[j,i] * x_i$ for all j such that $L_1[j,i] \neq 0$. At the end of updation by L_1, the size of vector b_1 is reduced to half its original size (see figure 3).

- *Step 2* : The reduced b_0 is copied to form vectors b_{00} and b_{01} whereas the reduced b_1 is copied to form vectors b_{10} and b_{11}. The new vectors b_{00} and b_{10} are modified by L_{00} and L_{10} respectively in forward direction whereas the vectors b_{01} and b_{11} are modified by L_{01} and L_{11} respectively in backward direction. Thus the size of these new b-vectors gets reduced by another factor of half (see figure 3).

- *Step 3* : This process of reducing the size of b-vectors and doubling their numbers continues for $\log N$ stages by which time there will be N b-vectors of only one element each. These N b-vectors, when divided by N elements obtained at the end of factorization phase, will give N x-vector elements.

3.2 Increasing Parallelism by Exploiting Sparsity

In the above scheme we observe that the process of modifying a b-vector through successive columns of a trapezoid is inherently sequential and is communication intensive in case the successive columns happen to reside on separate processors. George et.al. have proposed in [6], parallel schemes for solving sparse triangular systems resulting from regular Cholesky factorization. Their scheme is an adaptaion of the corresponding dense algorithm proposed by Romine and Ortega in [24] and it uses the following inner product form to carry out forward factorization.

$$x_i = \left(b_i - \sum_{\{j | L[i,j] \neq 0\}} (L[i,j] * x_j) \right) / L[i,i] \qquad i = 1, 2, \cdots, N$$

Since the columns and the corresponding solution components are distributed among the processors, the inner product computation is partitioned accordingly.

The above concept of distributed computation of inner product can be applied to the BS algorithm. Consider the case where the vector b_{x0} is to be updated by the trapezoid L_{x0} in

the forward direction. Instead of moving the vector b_{x0} from left to right across the trapezoid L_{x0}, each element $b_{x0}[i]$ is updated as follows. Each processor computes the products of the elements of the row i of the trapezoid that it contains with the corresponding elements of the solution vector x and sends their sum i.e., the partial inner product, to the processor containing column i. Upon receiving the contributions to the inner product from each processor, the processor storing the column i subtracts them from b_{x0}. If column i belongs to the first half of the matrix A_{x0} then, after subtracting the complete inner product of row i in L_{x0} from $b_{x0}[i]$, the processor storing the column i computes $x_i = b_{x0}[i]/L_{x0}[i,i]$. This x_i is then used for calculating the partial inner products of rows $j > i$. On the other hand if the column i belongs to the second half then after subtracting the complete inner product of row i in L_{x0} from $b_{x0}[i]$, two copies of the element $b_{xo}[i]$, namely $b_{x00}[i]$ and $b_{x01}[i]$, are made for modification at the next stage of the BS algorithm. Similar mechanism operates while updating a vector b_{x1} with a trapezoid L_{x1} in the backward direction. The complete details of the BS algorithm are given below.

Algorithm 2 (* The bidirectional substitution algorithm *)
begin
  for $s := 1$ **to** $\log N$ **do**
    parbegin
      Forward_modify($List_{myid}$,s);
      Backward_modify($List_{myid}$,s);
    parend
end

procedure Forward_modify($List$,s)
begin
  Let b_{x0} be the forward copy of the b-vector to be modified
    by trapezoid L_{x0} at stage s.
  for $i := 1$ **to** N **do** $t_i := 0$;
  for all $i \in List$ **do**
    for all j such that processor P_j has nonzeros belonging to row i of L_{x0} **do**
      receive message (i,t) having partial inner product t from processor P_j;
      $b_{x0}[i] := b_{x0}[i] - t$;
    if column i belongs to the first half of L_{x0} **then**
      $x_i := b_{x0}[i]/L_{x0}[i,i]$;
      for all j such that $L_{x0}[j,i] \neq 0$ **do**
        $t_j := t_j + x_i * L_{x0}[j,i]$;

if x_k has been calculated for all k such that $L_{x0}[j,k] \neq 0$ and
$k \in List$ **then**
send message (j,t_j) to processor storing column j;
else if $s < \log N$ **then**
$b_{x00}[i] := b_{x0}[i]$;
$b_{x01}[i] := b_{x0}[i]$;
else (* $s = \log N$ *) $x_i := b_{x0}[i]/L_{x0}[i]$;
end

procedure Backward_modify($List$,s)
begin
Let b_{x1} be the backward copy of the b-vector to be modified
by trapezoid L_{x1} at stage s.
for $i := 1$ **to** N **do** $t_i := 0$;
for all $i \in List$ **do**
for all j such that processor P_j has nonzeros belonging to row i of L_{x1} **do**
receive message (i,t) having partial inner product t from processor P_j;
$b_{x1}[i] := b_{x1}[i] - t$;
if column i belongs to the second half of L_{x1} **then**
$x_i := b_{x1}[i]/L_{x1}[i,i]$;
for all j such that $L_{x1}[j,i] \neq 0$ **do**
$t_j := t_j + x_i * L_{x1}[j,i]$;
if x_k has been calculated for all k such that $L_{x1}[j,k] \neq 0$ and
$k \in List$ **then**
send message (j,t_j) to processor storing column j;
else if $s < \log N$ **then**
$b_{x10}[i] := b_{x1}[i]$;
$b_{x11}[i] := b_{x1}[i]$;
else (* $s = \log N$ *) $x_i := b_{x1}[i]/L_{x1}[i]$;
end

As in the case of the BSCF algorithm, a special situation arises when $p = 2^d$ and $N = 2^n$ ($n, d \in \mathcal{N}$). After $d = \log p$ stages, the BSCF algorithm switches over to the regular sparse Cholesky factorization and produces triangular factor matrix of the form L_{ind} in the last stage such that $A_{ind} = L_{ind}L_{ind}^T$. Thus in the substitution phase, let b_{ind} be one of the p reduced vectors after $\log p$ stages of BS algorithm. We now switch over to the sequential substitution algorithm for solving the two triangular systems, $L_{ind}y = b_{ind}$ and $L_{ind}^T x = y$.

In this manner, we avoid executing excessive number of floating point operations when all the remaining computations are resricted to occur within individual processors.

In the next two sections, we describe the ordering and the symbolic factorization algorithms that precede the BSCF algorithm.

4 Ordering the Sparse Symmetric Matrix for Bidirectional Factorization

A good initial ordering of a sparse matrix A is crucial to the efficient solution of the sparse symmetric system $Ax = b$. The basic aim of the ordering phase is to reorder the columns of the coefficient matrix in such a manner that during the factorization phase, the amount of fill-in is minimized and the degree of parallelism is maximized. In a parallel environment, the former aim is not as important as the latter aim since large amounts of memory are available very cheaply. Various techniques for the ordering phase can be found in [3, 4, 5, 16, 22].

Sparse symmetric matrices chiefly arise from $k \times k$ regular grids that are encountered in finite element problems. The principal ordering heuristic used for reordering the matrices obtained from the regular grid problems is the popular *nested dissection* ordering method [4, 5]. The nested dissection ordering yields short and wide elimination trees that are well suited for parallel factorization algorithms. For regular Cholesky factorization, this ordering technique satisfies the criteria of both low fill-in and short and wide elimination trees. However, the nested dissection ordering in its existing form is not suited for the BSCF algorithm due to reasons given below. Recall that in section 2.2 we defined the concepts of forward elimination tree and backward elimination tree for the BSCF algorithm. The degree of parallelism while factorizing in forward direction depends on the shape of the forward elimination tree and that for factorizing in backward direction depends on the shape of the backward elimination tree. An ideal ordering for the BSCF algorithm is one in which both the elimination trees are as short and wide as possible. The forward elimination tree obtained from nested dissection algorithm is short and wide and hence desirable for parallel factorization. On the other hand the backward elimination tree obtained from nested dissection algorithm is lean and tall and hence undesirable for parallel factorization.

In the remaining part of this section, with the help of an example of a 7×7 grid, we show why the regular nested dissection algorithm is not suited for BSCF algorithm and then we describe how it can be modified to yield orderings suitable for the BSCF algorithm.

The nested dissection algorithm begins by recursively dividing a $k \times k$ grid into two disjoint parts using a set of nodes as *separator* nodes and applying the nested dissection algorithm

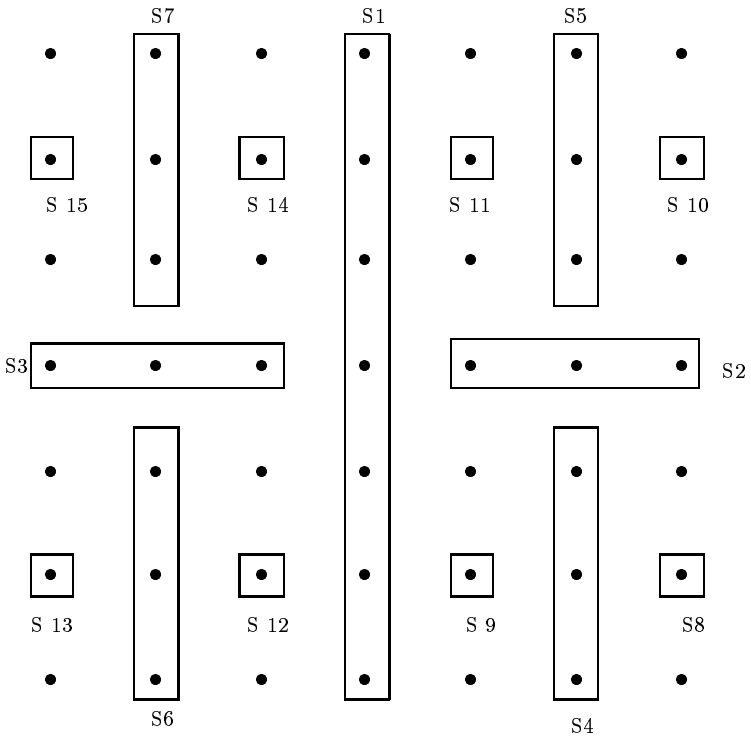


Figure 4: Dissection of a 7×7 grid by separators during nested dissection

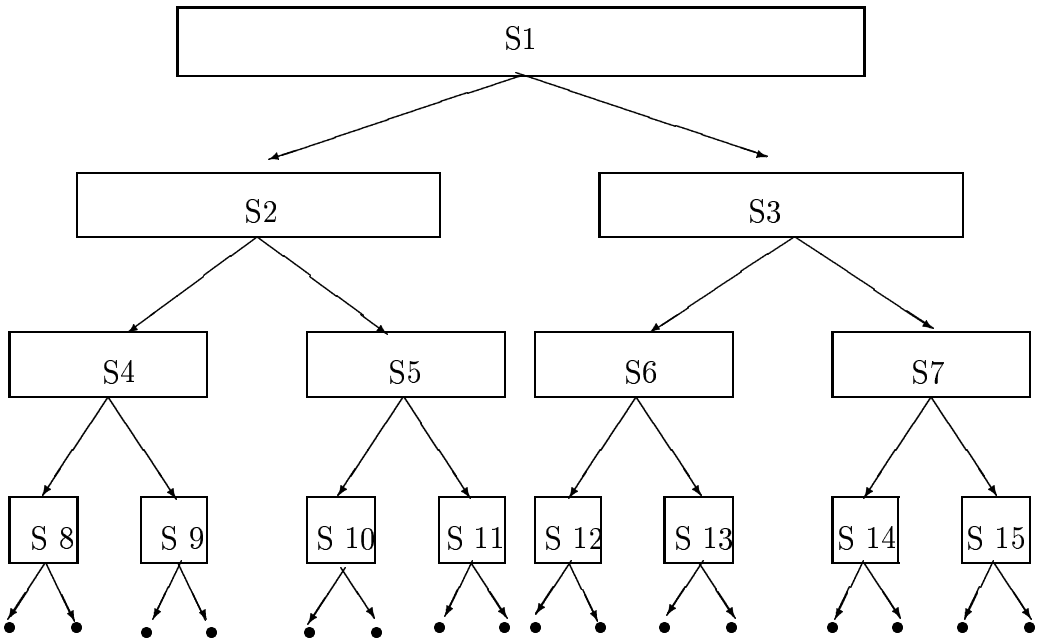


Figure 5: The nested dissection tree for a 7×7 grid

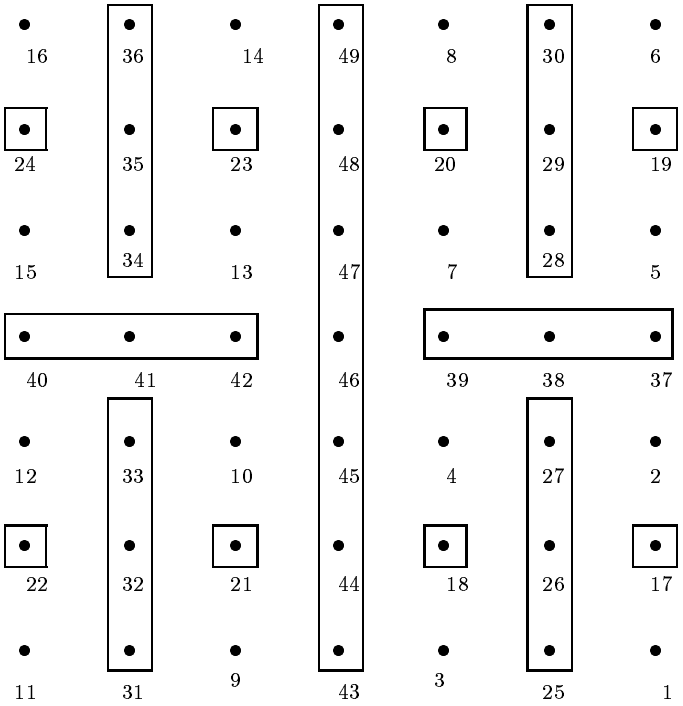


Figure 6: Ordering of a 7×7 grid using regular nested dissection ordering

again to the two separated halves. Figure 4 shows the manner in which the separators (S1 to S15) divide a 7×7 grid. The recursive division of the grid yields a tree structure of separators and nodes as shown in figure 5. We call this tree a *nested dissection tree*. The internal nodes of the tree are *separator blocks* and the leaves of the tree are blocks of node(s) at lowermost level which cannot be further sub-divided using nested dissection. The dimension of such blocks can be 1×1, 1×2, 2×1 or 2×2. Such indivisible blocks are called *leaf blocks*.

In regular nested dissection ordering, all the grid points at the leaf blocks(say at level 0) are numbered in ascending order. Then the separator grid points at level 1 are numbered, then level 2 and so on until the grid points at the root separator blocks get numbered. The ordering resulting from this scheme is shown in figure 6 and the forward and backward elimination trees resulting from this ordering are shown in figure 7. As seen from figure 7, although the forward tree is short and wide, the backward tree is lean and tall. Hence this ordering is not conducive for good performance of the BSCF algorithm.

We now look at a modification of the regular nested dissection algorithm which produces orderings that provide reasonably good parallelism properties in both forward and backward directions. We call this heuristic as the *bidirectional nested dissection ordering* which proceeds as follows.

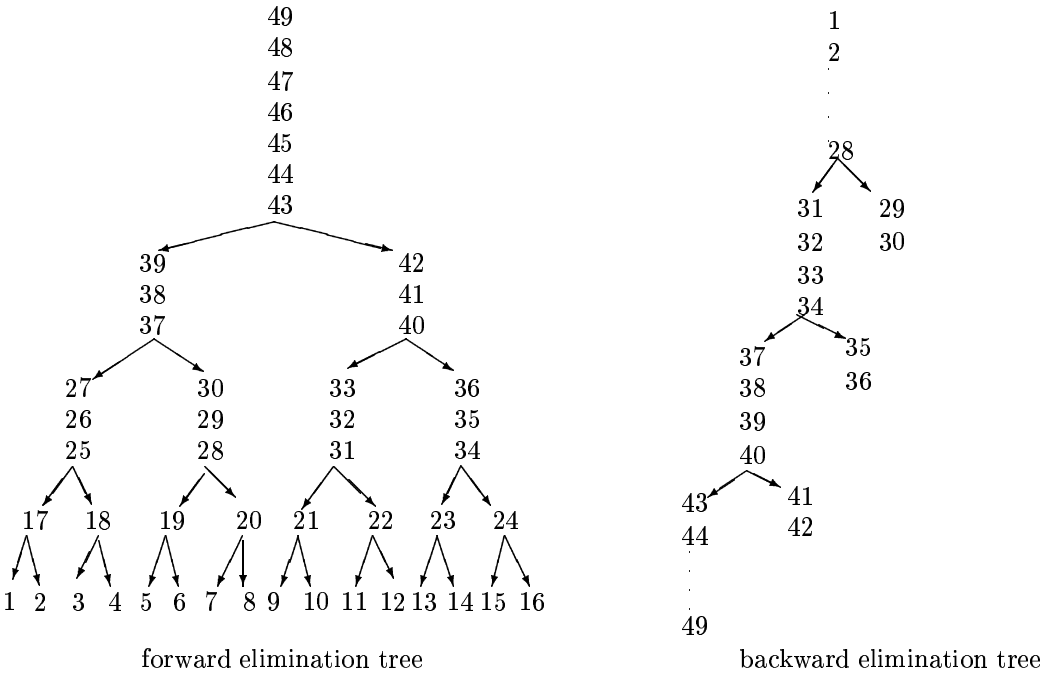


Figure 7: The forward and backward elimination trees for a 7×7 grid obtained using regular nested dissection ordering

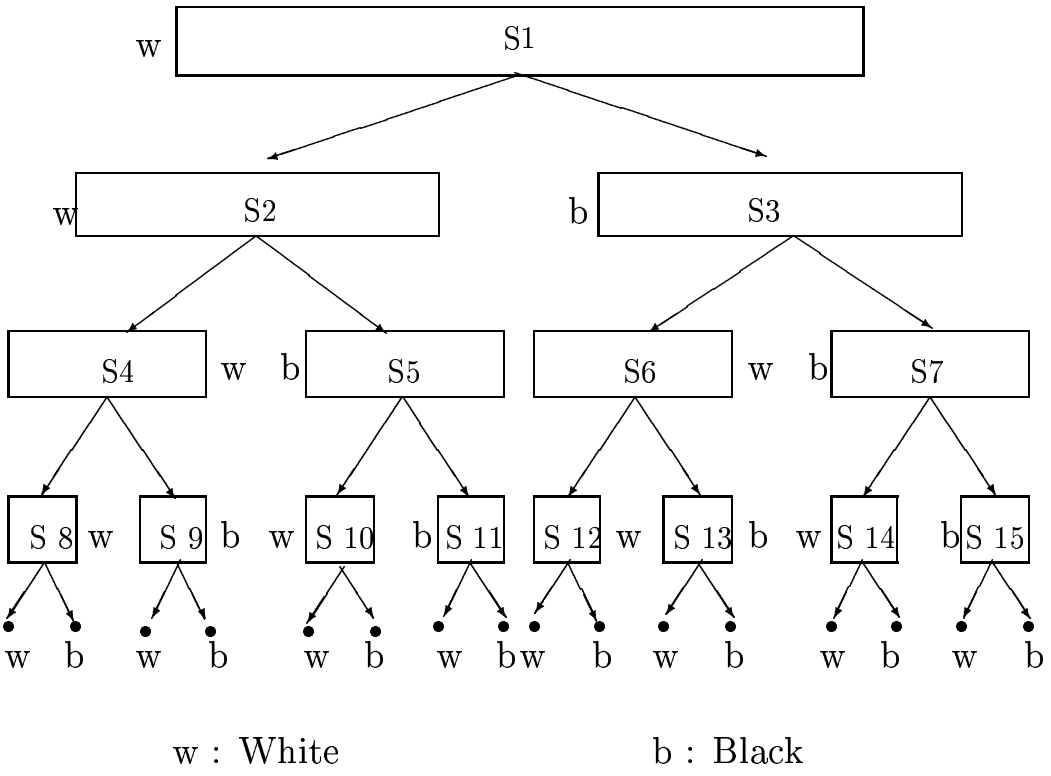


Figure 8: The colouring of tree nodes in bidirectional nested dissection ordering

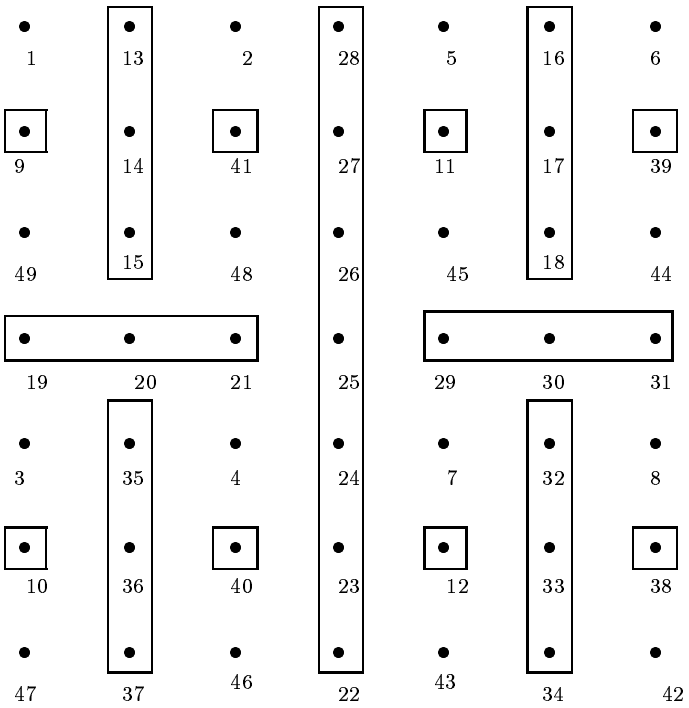
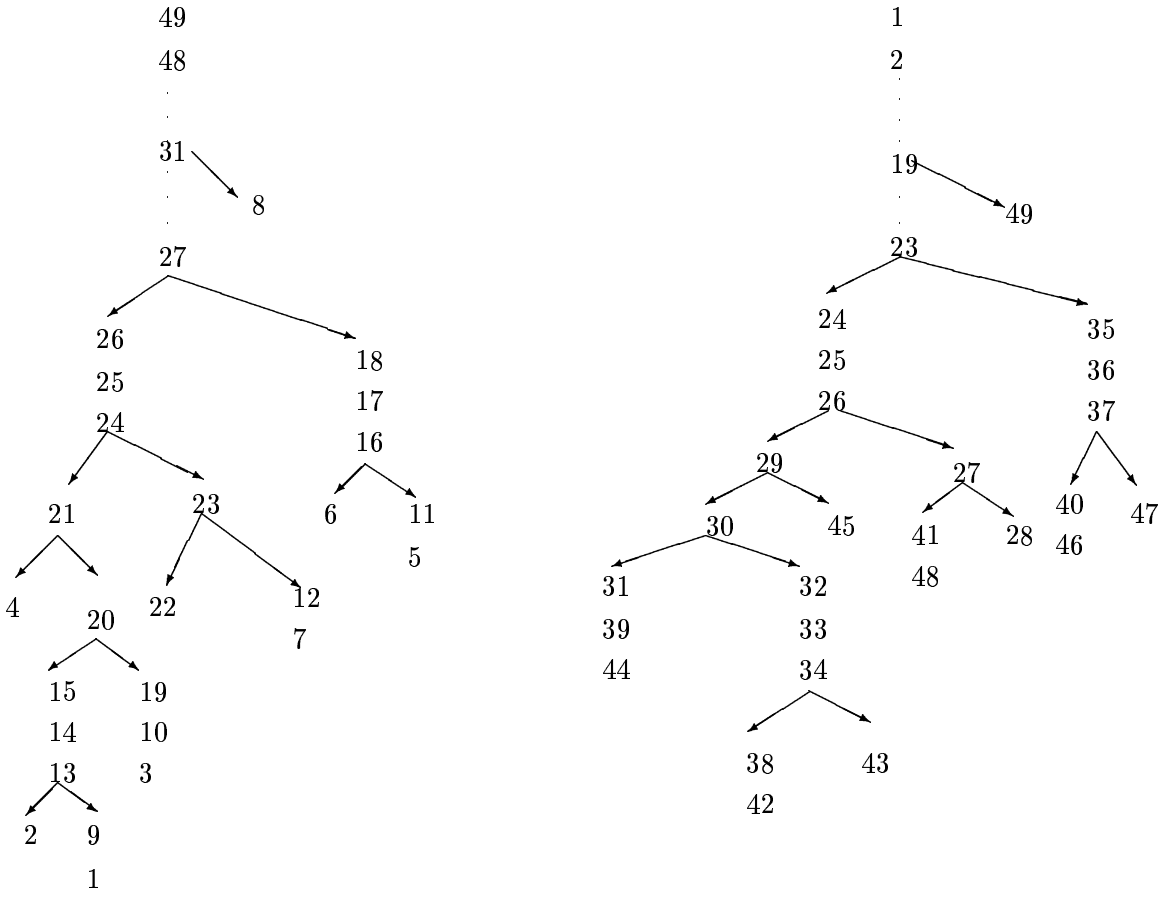


Figure 9: Ordering of a 7×7 grid using bidirectional nested dissection ordering

- *Step 1* : Carry out the dissection part of the nested dissection algorithm as described above. This gives a nested dissection tree as shown in figure 5.
- *Step 2* : At each level of the nested dissection tree, label approximately half of the tree nodes as white and the other half as black, as shown in figure 8.
- *Step 3* : While numbering the grid points, proceed as follows.
  1. Keep two counts - $whiteCount$ initialized to 1 and $blackCount$ initialized to $k \times k$.
  2. Take a grid point at level 0. If the leaf node to which it belongs is white then number the grid point as $whiteCount$ and increment $whiteCount$. Otherwise the leaf node is black. Hence number the grid point as $blackCount$ and decrement $blackcount$.
  3. Apply the above step to all grid points of each node at level 0 followed by each node at level 1 and so on upto the root.

The ordering obtained from this scheme is shown in figure 9 and the corresponding forward and backward elimination trees are shown in figure 10. As seen in this figure, although the forward elimination tree is not as short and wide as in the case of regular nested dissection



forward elimination tree

backward elimination tree

Figure 10: The forward and backward elimination trees for a 7×7 grid obtained using bidirectional nested dissection ordering

ordering, the backward tree is definitely more conducive for good performance of parallel factorization than in the previous case. Essentially we have succeeded in balancing the degree of parallelism in both forward and backward directions so that lack of parallelism in any one direction does not act as a bottleneck to the entire BSCF algorithm.

In the next section we look at the bidirectional symbolic factorization algorithm which allocates memory and sets up the appropriate data structures prior to the BSCF algorithm.

5 The Bidirectional Symbolic Factorization Algorithm

The principal aim of the symbolic factorization phase is to determine apriori, the data structure of the factor matrices that result from the numerical factorization phase. As seen in section 2, the BSCF algorithm creates a series of trapezoidal factor matrices of multipliers. Hence, the bidirectional symbolic factorization algorithm, which precedes the BSCF phase, does the following.

- It determines the structure of each trapezoidal factor matrix at each of the $\log N$ stages and
- It initializes the data structures for the sets $F_i^{(s)}$ and $B_i^{(s)}$ which are required during the BSCF algorithm.

We define $Colstruct(A_{x0}, i)$ to denote the set of row indices of nonzeros in the sub-diagonal part of column i in the forward matrix A_{x0}.

$$Colstruct(A_{x0}, i) = \{j \mid j > i \text{ and } A_{x0}[j, i] \neq 0\}.$$

In a similar fashion, we define $Colstruct'(A_{x1}, i)$ to denote the set of row indices of nonzeros in the super-diagonal part of column i of the backward matrix A_{x1}.

$$Colstruct'(A_{x1}, i) = \{j \mid j < i \text{ and } A_{x1}[j, i] \neq 0\}.$$

We now describe the bidirectional symbolic factorization algorithm.
Algorithm 3 (*The bidirectional symbolic factorization algorithm*)
begin
  for $s := 1$ **to** $\log N$ **do**
    for $col := 1$ **to** N **do**
      $F_{col}^{(s)} := \phi; B_{col}^{(s)} := \phi;$
  for $s := 1$ **to** $\log N$ **do**
    for $col := 1$ **to** N **do**

Forward_SF(col,s);

for $col := N$ **downto** 1 **do**

Backward_SF(col,s);

end

procedure Forward_SF(col,s)

begin

Let A_{x0} be the forward sub-matrix that contains column col at stage s;

if col belongs to the first half of A_{x0} **then**

Calculate $fparent(col, A_{x0})$ using definition given in section 3.2.2;

if $fparent(col, A_{x0})$ belongs to the first half of A_{x0} **then**

$Colstruct(A_{x0}, fparent(col, A_{x0})) :=$

$Colstruct(A_{x0}, fparent(col, A_{x0}) \cup Colstruct(A_{x0}, col)$;

for all j such that j belongs to second half of A_{x0} and $A_{x0}[col, j] \neq 0$ **do**

$Colstruct(A_{x0}, j) := Colstruct(A_{x0}, j) \cup Colstruct(A_{x0}, col)$;

for all j such that $j \in Colstruct(A_{x0}, col)$ **do**

$F_j^{(s)} := F_j^{(s)} \cup \{col\}$;

else

$Colstruct(A_{x00}, col) := Colstruct(A_{x0}, col)$;

for all $j \in Colstruct(A_{x0}, col)$ **do**

$Colstruct'(A_{x01}, j) := Colstruct'(A_{x01}, j) \cup \{col\}$;

end

procedure Backward_SF(col,s)

begin

Let A_{x1} be the backward sub-matrix that contains column col at stage s;

if col belongs to the second half of A_{x1} **then**

Calculate $bparent(col, A_{x1})$ using definition given in section 3.2.2;

if $bparent(col, A_{x1})$ belongs to the second half of A_{x1} **then**

$Colstruct'(A_{x1}, fparent(col, A_{x1})) :=$

$Colstruct'(A_{x1}, fparent(col, A_{x1}) \cup Colstruct'(A_{x1}, col)$;

for all j such that j belongs to first half of A_{x1} and $A_{x1}[col, j] \neq 0$ **do**

$Colstruct'(A_{x1}, j) := Colstruct'(A_{x1}, j) \cup Colstruct'(A_{x1}, col)$;

for all j such that $j \in Colstruct'(A_{x1}, col)$ **do**

$B_j^{(s)} := B_j^{(s)} \cup \{col\}$;

else

for all $j \in Colstruct'(A_{x1}, col)$ **do**

$Colstruct(A_{x10}, j) := Colstruct(A_{x10}, j) \cup \{col\}$;

$Colstruct'(A_{x11}, col) := Colstruct'(A_{x1}, col)$;

end

The bidirectional symbolic factorization algorithm described above has time complexity proportional to the number of nonzero elements stored in trapezoids at each stage. Since the symbolic factorization algorithm is executed only once while solving for multiple b-vectors and also since this phase takes significantly lower time than the numerical factorization phase, parallelizing this phase does not yield significant improvements in the overall performance.

For the case of regular symbolic factorization, parallel algorithms have been described in [8, 10, 13]. While the former scheme by George et.al. requires the information about the elimination tree structure apriori, the latter scheme by Kumar et.al. does not require this information and uses the concept of *false elimination trees (fet)* to compute the symbolic factorization. More specifically, the computation begins with the leaves of the false elimination tree which pass their column structure information to their true parents. Each internal node then combines the column structures of all its children with its own column structure, computes the true parent and sends its column structure information to its true parent. This process continues till all the information propagates to the root node.

We have developed a parallel bidirectional symbolic factorization algorithm based on a similar concept of *forward* and *backward* false elimination trees.

- $ffparent(i, s)$ denotes the false forward parent of a column i in the sub-matrix A_{x0} being factorized in the forward direction at stage s.

$$ffparent(i, s) = min\{j \mid j \in \text{ first half of } A_{x0} \text{ and } j \in Colstruct(A_{x0}, i)\}.$$

- $fbparent(i, s)$ denotes the false backward parent of a column i in the sub-matrix A_{x1} being factorized in the backward direction at stage s.

$$fbparent(i, s) = max\{j \mid j \in \text{ second half of } A_{x1} \text{ and } j \in Colstruct'(A_{x1}, i)\}.$$

The details of this algorithm are described below.

Algorithm 4 (*The parallel bidirectional symbolic factorization*)

begin

for $s := 1$ to $\log N$ **do**

parbegin

```
        Forward_SF(List_myid,s);
        Backward_SF(List_myid,s);
        parend
end.

procedure Forward_SF(List,s)
begin
    for each i ∈ List do
        Let A_x0 be the forward sub-matrix to which column i belongs at stage s;
        dummy_parent := last node of sub-matrix A_x0;
        Determine the false forward parent ffparent(i, s);
        send ffparent(i, s) to processor containing dummy_parent;
        if i = dummy_parent then
            receive ffparent(j, s) from each column j;
            broadcast forward fet T_ff constructed from received
                ffparent information;
        receive forward fet T_ff broadcast from dummy_parent;
        Let the children of column i in T_ff be CHLD(i);
        (*initialise the expected and accumulated weights for node i*)
        exp_wt(i) :=| CHLD(i) |; acc_wt(i) := 0;
        first(i):=true;
        if column i is a true leaf of T_ff and column i is in
            first half of sub-matrix A_x0 then
            send Colstruct(A_x0, i) to ffparent(i, s) with weight 1;
            send Colstruct(A_x0, i) with weight 0 to all nodes j in second half of
                A_x0 such that j ∈ Colstruct(A_x0, i) ;
    repeat
        receive a message S intended for column i;
        Let the message be from processor storing column j with weight w;
        if column i is in first half of sub-matrix A_x0 then
        case type of S
                attach or ordinary:
                    Colstruct(A_x0, i) := Colstruct(A_x0, i) ∪ Colstruct(A_x0, j);
                    acc_wt := acc_wt + w;
                    if j ∈ CHLD(i) then delete j from CHLD(i);
                    if (| CHLD(i) = 0 |) and (acc_wt(i) ≥ exp_wt(i)) then
```

```
                    ffparent(i, s) := k where k = min(Colstruct(A_x0, i));
                    if ffparent(i) has changed then
                        send a detach message to old parent;
                    if first(i) then
                        wt := acc_wt(i) − exp_wt(i) + 1;
                        exp_wt(i) := 0;
                        first(i) :=false;
                    else
                        wt := w;
                    send Colstruct(A_x0, i) to ffparent(i) with weight wt;
                    send Colstruct(A_x0, i) to all nodes j in second half of A_x0
                        such that j ∈ Colstruct(A_x0, i) with weight 0;
                detach :
                    delete j from CHLD_(i);
        else
            case type of S
                attach or ordinary:
                    if j ∈ Colstruct(A_x0, i) then
                        Colstruct(A_x0, i) := Colstruct(A_x0, i) ∪ Colstruct(A_x0, j);
                detach:
                    if i = dummy_parent then
                        delete j from CHLD(i);
                        if (| CHLD(i) = 0 |) then
                            broadcast forward phase over message;
    until S is forward phase over message;
    for each i ∈ List do
        if column i is in second half of sub-matrix then
            Colstruct(A_x00, i) := Colstruct(A_x0, i);
            for all j such that A_x0[j, i] ≠ 0 do
                Colstruct'(A_x01, j) := Colstruct(A_x01, j) ∪ i;
end

procedure Backward_SF(List,s)
begin
    for each i ∈ List do
        Let A_x1 be the backward sub-matrix to which column i belongs at stage s;
```

dummy_parent := last node of sub-matrix A_{x1};
Determine the false backward parent $fbparent(i, s)$;
send $fbparent(i, s)$ to processor containing *dummy_parent*;
if $i = dummy\_parent$ **then**
receive $fbparent(j, s)$ from each column j;
broadcast backward fet T_{fb} constructed from received
$fbparent$ information;
receive backward fet T_{fb} broadcast from *dummy_parent*;
Let the children of column i in T_{fb} be $CHLD(i)$;
$exp\_wt(i) :=\mid CHLD(i) \mid$; $acc\_wt(i) := 0$;
$first(i)$:=*true*;
if column i is a true leaf of T_{fb} **and** column i is in second half
of sub-matrix A_{x1} **then**
send $Colstruct'(A_{x1}, i)$ to $fbparent(i, s)$ with weight 1;
send $Colstruct'(A_{x1}, i)$ with weight 0 to all nodes j in first half
of sub-matrix A_{x1} such that $j \in Colstruct'(A_{x1}, i)$;
repeat
receive a message S intended for column i;
Let the message be from processor storing column j with weight w;
if column i is in second half of sub-matrix A_{x1} **then**
case type of S
attach or ordinary:
$Colstruct'(A_{x1}, i) := Colstruct'(A_{x1}, i) \cup Colstruct'(A_{x1}, j)$;
$acc\_wt := acc\_wt + w$;
if $j \in CHLD(i)$ **then** delete j from $CHLD(i)$;
if $(\mid CHLD(i) = 0 \mid)$ **and** $(acc\_wt(i) \geq exp\_wt(i))$ **then**
$fbparent(i, s) := k \; where \; k = max(Colstruct'(A_{x1}, i))$;
if $fbparent(i)$ has changed **then**
send a detach message to old parent;
if $first(i)$ **then**
$wt := acc\_wt(i) - exp\_wt(i) + 1$;
$exp\_wt(i) := 0$;
$first(i)$:=**false**;
else
$wt := w$;
send $Colstruct'(A_{x1}, i)$ to $fbparent(i)$ with weight wt;

send $Colstruct'(A_{x1}, i)$ to all nodes j in first half of sub-matrix
such that $j \in Colstruct'(A_{x1}, i)$ with weight 0;
detach :
delete j from $CHLD_(i)$;
else
case type of S
attach or ordinary:
if $j \in Colstruct'(A_{x1}, i)$ **then**
$Colstruct'(A_{x1}, i) := Colstruct'(A_{x1}, i) \cup Colstruct'(A_{x1}, j)$;
detach:
if $i = dummy\_parent$ **then**
delete j from $CHLD(i)$;
if ($\mid CHLD(i) = 0 \mid$) **then**
broadcast backward phase over message;
until S is backward phase over message;
for each $i \in List$ **do**
if column i is in first half of sub-matrix **then**
for all j such that $A_{x1}[j, i] \neq 0$ **do**
$Colstruct(A_{x10}, j) := Colstruct'(A_{x10}, j) \cup i$;
$Colstruct'(A_{x11}, i) := Colstruct'(A_{x1}, i)$;
end

6 Experimental Results and Performance Analysis

To evaluate the performance of the entire bidirectional scheme presented in this paper, we implemented a hypercube simulator in C language and compared the *speedups* obtained from the bidirectional scheme with those obtained from the regular scheme. We used SPARC Classic machines to carry out our simulations.

In the bidirectional scheme, we implemented each of the four phases as follows.

- *Ordering* : The bidirectional nested dissection ordering described in section 4.
- *Symbolic factorization* : The sequential bidirectional symbolic factorization algorithm described in section 5.
- *Numerical factorization* : The parallel BSCF algorithm described in section 2.

- *Substitution* :The parallel BS algorithm described in section 3.

In the regular scheme, we implemented each of the four phases as follows.

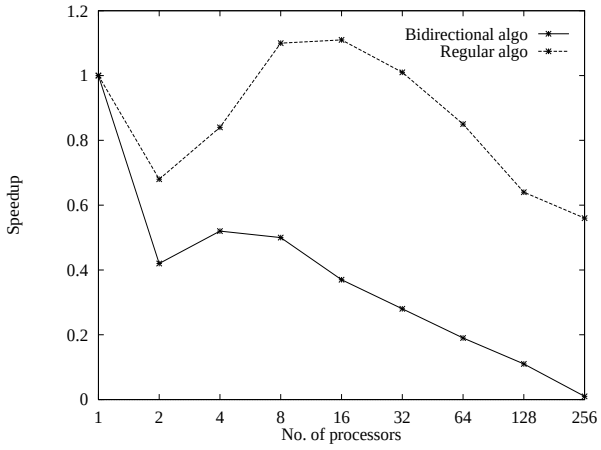
- *Ordering* : The regular nested dissection algorithm for ordering a $k \times k$ grid [4].
- *Symbolic factorization* : The sequential symbolic factorization algorithm presented in [8].
- *Numerical factorization* : The parallel fan-in algorithm given in [1].
- *Substitution* :The elimination tree based forward and back substitution algorithms given in [14].

Mapping of columns onto processors is an important issue. For the bidirectional scheme, we have used the *block wrap around mapping* using gray code [23] whereas for the regular algorithm we have used the *subtree-to-processor* mapping [9] based on elimination tree.

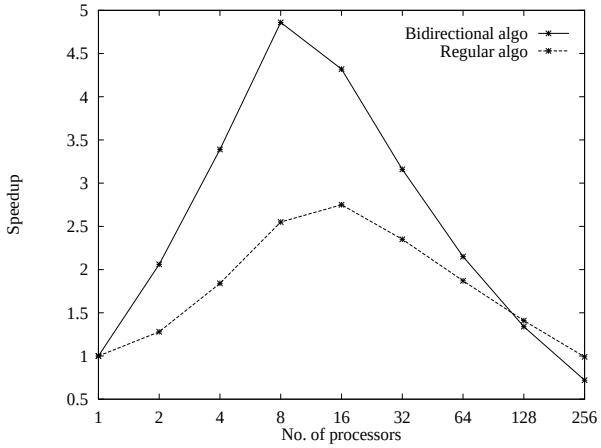
The parameters that were varied were the grid size k(16 and 32), the number of processors p(1 to 1024), the number of b-vectors for which solution vector x was obtained, and the C/E ratio i.e., the ratio of time for communicating a floating point data between two neighbouring processors to the time for a floating point operation(50 and 100). Figures 11, 12, 13, and 14 show the comparison of the measured speedups of the two schemes for various values of the above parameters.

As mentioned earlier in section 1, the first three phases, namely ordering, symbolic factorization, and numerical factorization, are executed only once and the substitution phase is repeatedly executed for each one of the different b-vectors.The output of the factorization phase of the bidirectional algorithm is a series of trapezoidal factor matrices whereas the output of the regular factorization algorithm is a pair of lower and upper triangular factor matrices. As a result, the inputs to the substitution phase of bidirectional and regular algorithms also differ. For separate comparison of the two phases of bidirectional and regular algorithms, we have considered a pseudo-speedup ratio for the bidirectional algorithm. This is a ratio of the time taken by the best sequential regular algorithm for the factorization (substitution) phase to the time taken by the parallel bidirectional algorithm for the factorization (substitution) phase.

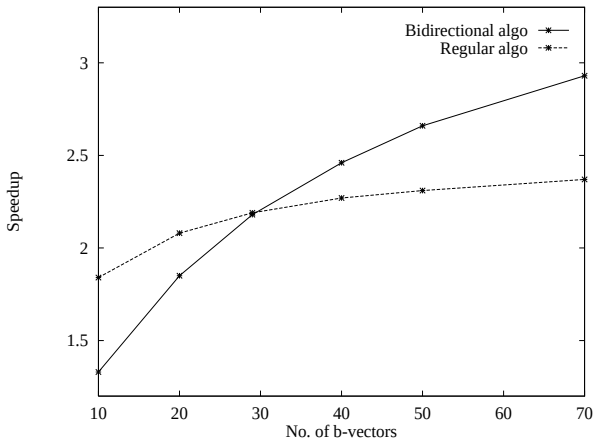
Therefore figures 11(a), 12(a), 13(a), and 14(a) compare the pseudo-speedup of the bidirectional algorithm with the speedup of the regular algorithm for the first three phases put together. The figures 11(b), 12(b), 13(b), and 14(b) compare the pseudo-speedup of the bidirectional algorithm with the speedup of the regular algorithm for the substitution phase alone. Figures 11(c), 12(c), 13(c), and 14(c) plot the actual speedups of bidirectional and



(a) factorization

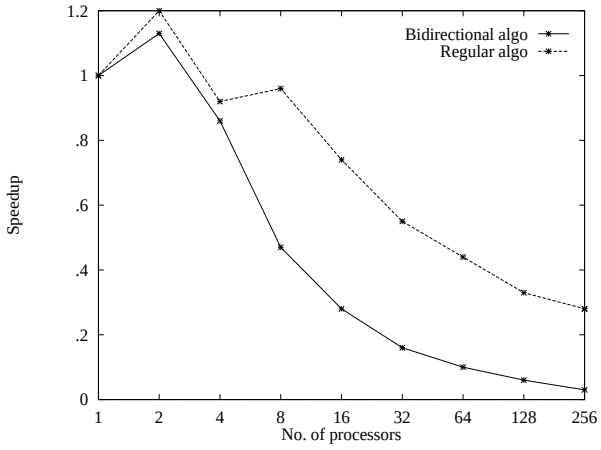


(b) substitution

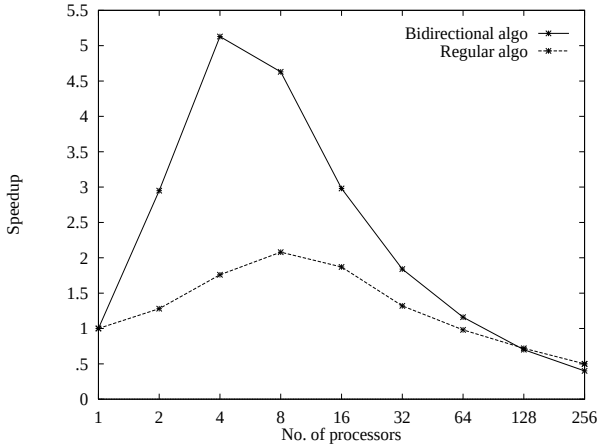


(c) solving multiple b-vectors with 8 processors

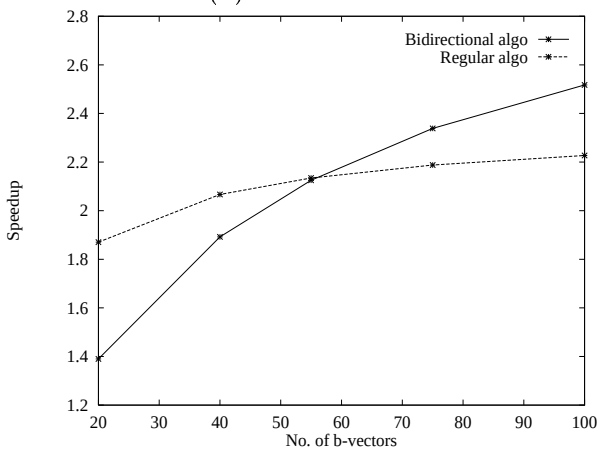
Figure 11: Speedups obtained for bidirectional algorithm versus regular algorithm for a 16×16 grid (i.e., $N = 256$) with $C/E = 50$



(a) factorization

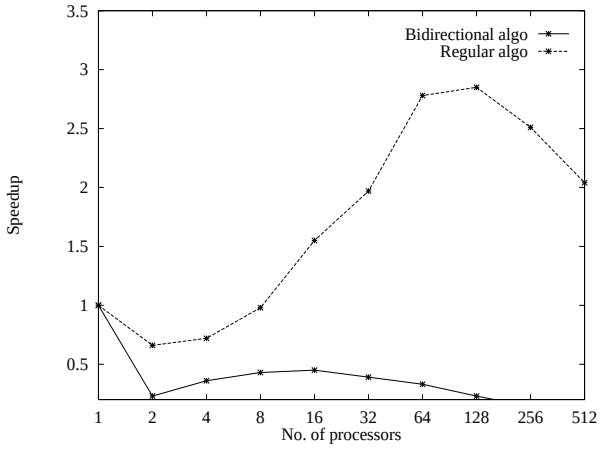


(b) substitution

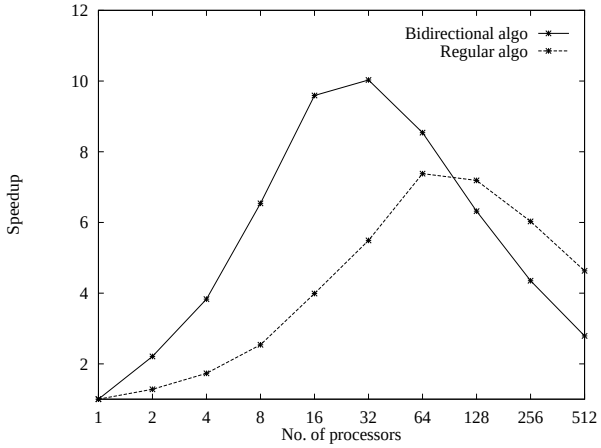


(c) solving multiple b-vectors with 8 processors

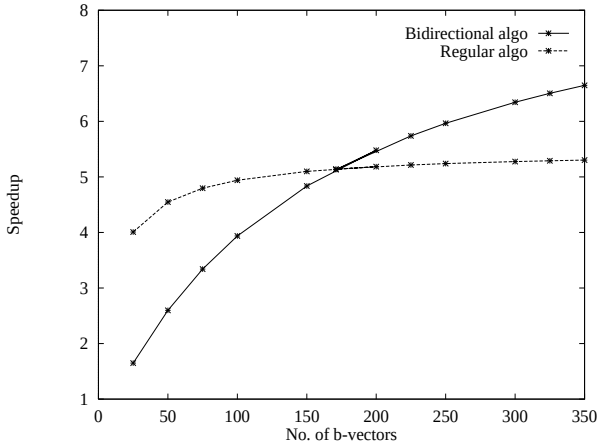
Figure 12: Speedups obtained for bidirectional algorithm versus regular algorithm for a 16×16 grid (i.e., $N = 256$) with $C/E = 100$



(a) factorization

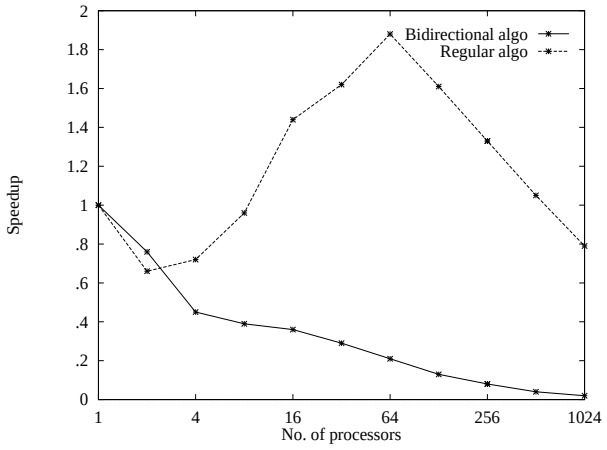


(b) substitution

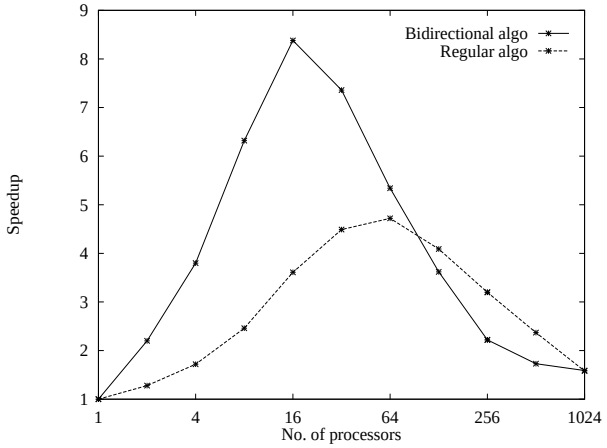


(c) solving multiple b-vectors with 8 processors

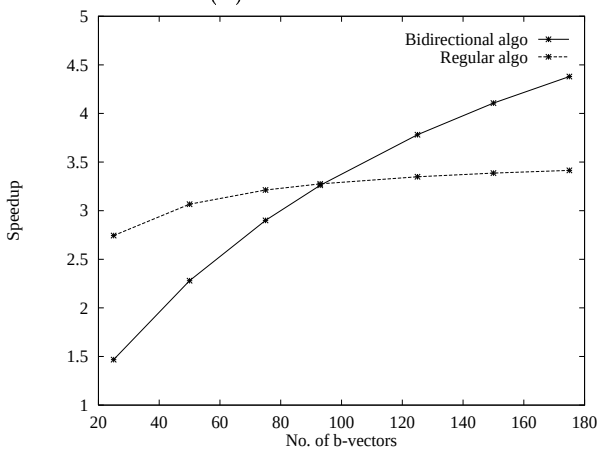
Figure 13: Speedups obtained for bidirectional algorithm versus regular algorithm for a 32×32 grid (i.e., $N = 1024$) with $C/E = 50$



(a) factorization



(b) substitution



(c) solving multiple b-vectors with 16 processors

Figure 14: Speedups obtained for bidirectional algorithm versus regular algorithm for a 32×32 grid (i.e., $N = 1024$) with $C/E = 100$

regular algorithms for all the four phases put together versus the number of b-vectors for which substitution phase is repeatedly executed. In figure 11(c), this comparison has been shown for the case when $p = 8$ and $k = 16$ (or $N = 256$) since, for $k = 16$, bidirectional factorization phase gives maximum speedup at $p = 8$. Similarly, in figure 12(c) $p = 8$ and $k = 16$, in figure 13(c) $p = 32$ and $k = 32$, and in figure 14(c) $p = 16$ and $k = 32$ (or $N = 1024$). These figures clearly indicate that with increasing number of b-vectors, the speedup obtained from our bidirectional scheme becomes higher than that obtained from the regular scheme. On increasing the problem size from $k = 16$ to 32, we observe that the magnitude of speedup obtained also increases. Increasing the C/E ratio causes a decrease in the magnitude of speedup obtained.

7 Conclusions

In this paper, we have proposed a new bidirectional algorithm for direct solution of sparse symmetric system of linear equations. This scheme generates a series of trapezoidal factor matrices during the factorization phase due to which the substitution phase has only one forward substitution component and, unlike the regular substitution algorithms, it does not possess a back substitution component. Thus the bidirectional algorithm is well suited for situations where the system of equations has to be solved for multiple b-vectors. We have demonstrated the effectiveness of the bidirectional algorithm by comparing it with the regular methods for solving sparse symmetric systems.

In Part II of this paper, we present the bidirectional algorithm for solution of general sparse linear systems. We describe the important differences with the symmetric coefficient matrix case, that arise in the ordering technique, the symbolic factorization phase, and message passing during numerical factorization phase.

Further work is possible in the direction of increasing the amount of parallelism in the factorization and substitution phases of the bidirectional algorithm. In this work, we have considered a situation where computations on a particular column, say i, for both forward and backward factorizations are handled by the same processor. However, the computations for forward and backward factorizations are independent of each other (i.e., concurrent) at every stage s. Same is the case with the computations on a column i in substitution phase. This concurrency has not been sufficiently exploited in the present work. In place of using p processors, we can use $2p$ processors, such that two processors are responsible for computations on each column - one handling computations related to forward factorization and the other related to backward factorization. Developing such a scheme is an open problem.

Acknowledgements This work is supported by the Indian National Science Academy, and the Department of Science and Technology.

References

[1] C.Ashcraft, S.C.Eisenstat and J.W.H.Liu, *A fan-in algorithm for distributed sparse numerical factorization*, SIAM J. Sci. Stat. Comput., Vol. 11, No. 3, 1990, pp. 593-599.

[2] C.Ashcraft, S.C.Eisenstat, J.W.H.Liu, and A.H.Sherman, *A comparison of three column based distributed sparse factorization schemes*, Technical Report YALEU/DCS/RR-810, Yale University, New haven, CT, 1990.

[3] J.M.Conroy, *Parallel nested dissection*, Parallel Computing, Vol. 16, 1990, pp. 139-156.

[4] A.George, *Nested dissection of a regular finite element mesh*, SIAM J. Numer. Anal., Vol. 10, No. 2, 1973, pp. 345-363.

[5] A.George, M.T.Heath, J.W.H.Liu and E.Ng, *Computer Solution of Large Sparse Positive Definite Systems* Prentice Hall, Englewood Cliffs, NJ, 1981.

[6] A.George, M.T.Heath, J.W.H.Liu and E.Ng, *Solution of sparse positive definite systems on hypercube*, J. Comput. Applied Math., Vol. 27, 1989, pp. 129-156.

[7] A.George, M.T.Heath, J.W.H.Liu and E.Ng, *Sparse Cholesky factorization on a local memory multiprocessor*, SIAM J. Sci. Stat. Comput., Vol. 9, No. 2, 1988, pp. 327-340.

[8] A.George, M.T.Heath, E.Ng and J.W.H.Liu, *Symbolic Cholesky factorization on local memory multiprocessor*, Parallel Computing, Vol. 5, 1987, pp. 85-95.

[9] A.George, J.W.H.Liu and E.Ng, *Communication results for parallel sparse Cholesky factorization on hypercube*, Parallel Computing, Vol. 10, No. 3, 1989, pp. 287-298.

[10] J.R.Gilbert and H.Hafsteinsson, *Parallel symbolic factorization for sparse linear systems*, Parallel Computing, Vol. 14, 1990, pp. 151-162.

[11] M.T.Heath, E.Ng and B.W.Peyton, *Parallel algorithms for sparse linear systems*, SIAM Review, Vol. 33, 1991, pp. 420-460.

[12] C.W.Ho, *Fast Parallel Algorithms Related to Chordal Graphs*, Ph.D. Thesis, Institute of Computer and Decision Sciences, National Tsing Hua University, Hsinchu, Taiwan, Republic of China, 1988.

[13] P.S.Kumar, M.K.Kumar and A.Basu, *A parallel algorithm for elimination tree computation and symbolic factorization*, Parallel Computing, Vol. 18, 1992, pp. 849-856.

[14] P.S.Kumar, M.K.Kumar and A.Basu, *Parallel algorithms for sparse triangular system solution*, Parallel Computing, Vol. 19, 1993, pp. 187-196.

[15] V.Kumar, A.Grama, A.Gupta, and G.Karypis, *Introduction to Parallel Computing - Design and Analysis of Algorithms*, Benjamin/Cummings Publishing Company Inc., 1994.

[16] C.E.Leiserson and T.G.Lewis, *Orderings for parallel sparse symmetric factorization*, In Proceedings of the Third SIAM Conference on Parallel Processing for Scientific Computing, 1987, pp. 27-32.

[17] G.Li and T.F.Coleman, *A new method for solving triangular systems on distributed memory message passing multiprocessors*, SIAM J. Sci. Stat. Comput., Vol. 10, 1989, pp. 382-396.

[18] J.W.H.Liu, *Computational models and task scheduling for parallel sparse Cholesky factorization*, Parallel Computing, Vol. 3, 1986, pp. 327-342.

[19] K.N.B.Murthy, *New Algorithms for Parallel Solution of Linear Equations on Distributed Memory Multiprocessors*, Ph.D. Thesis, Department of Computer Science and Engineering, Indian Institute of Technology, Madras, India, 1995.

[20] K.N.B.Murthy and C.S.R.Murthy, *A new Gaussian elimination based algorithm for parallel solution of linear equations*, Computers and Mathematics with Applications, Vol. 29, No. 7, 1995, pp. 39-54.

[21] E.Ng, *Parallel direct solution of sparse linear systems*, Parallel Supercomputing: Methods, Algorithms and Applications, John Wiley and Sons Ltd., 1989.

[22] F.Peters, *Parallel pivoting algorithms for sparse symmetric matrices*, Parallel Computing, Vol. 1, 1984, pp. 99-110.

[23] E.M.Reingold, J.Nievergelt, and N.Deo, *Combinatorial Algorithms : Theory and Practice*, Prentice Hall, Englewood Cliffs, NJ, 1977.

[24] C.H.Romine and J.M.Ortega, *Parallel solution of triangular systems of equations*, Parallel Computing, Vol. 6, 1988, pp. 109-111.